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Bonnesen's inequality for John domains in $\mathbb{R}^n$

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Abstract

We prove sharp quantitative isoperimetric inequalities for John domains in $\mathbb{R}^n$. We show that the Bonnesen-style inequalities hold true in $\mathbb{R}^n$ under the John domain assumption which rules out cusps. Our main tool is a proof of the isoperimetric inequality for symmetric domains which gives an explicit estimate for the isoperimetric deficit. We use the sharp quantitative inequalities proved in Fusco et al. (2008) [7] and Fuglede (1989) [4] to reduce our problem to symmetric domains.

Keywords: Quantitative isoperimetric inequality; John domain; Quasiconformal map

1. Introduction

The sharp isoperimetric inequality states that

$$n\alpha_n^{1/n}|E|^{(n-1)/n} \leqslant P(E) \tag{1.1}$$

always holds for Borel sets $E \subset \mathbb{R}^n$, $n \geqslant 2$, with finite n-measure $|E|$. Here $P(E)$ is the $(n-1)$-measure of the boundary (the distributional perimeter, see Section 2), and $\alpha_n = |B^n|$. Equality holds in (1.1) if and only if E is an n-ball (up to a set of measure zero). The quantitative isoperimetric inequalities are estimates which improve the latter statement: if (1.1) is almost an equality

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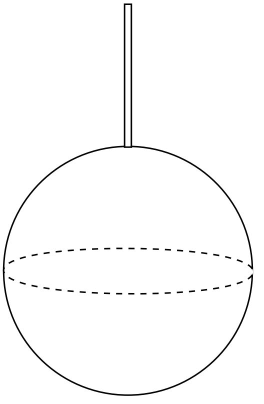

Fig. 1. Small isoperimetric deficit, large distortion.

for a set E, then E is almost a ball with respect to some geometric quantity. A classical estimate of this type is Bonnesen's inequality

$$\ell(\partial\Omega)^2 - 4\pi|\Omega| \geqslant 4\pi(R-s)^2, \tag{1.2}$$

valid for Jordan domains $\Omega \subset \mathbb{R}^2$. Here R and s are the circumradius and inradius of Ω, respectively. There is a large collection of similar inequalities for planar Jordan domains. Such inequalities are called Bonnesen-style inequalities, see [11].

Bonnesen's inequality does not hold for general domains in dimensions higher than two. This can be seen, for example, by gluing long, thin cusps to the unit n-ball. (See Fig. 1.) Recently, however, important quantitative isoperimetric inequalities for general Borel sets have been obtained in all dimensions. Let $E \subset \mathbb{R}^n$ be a Borel set and r its volume radius, that is, $|E| = |B^n(r)|$ for a ball $B^n(r)$. The isoperimetric deficit $\delta(E)$ is

$$\delta(E) = \frac{P(E)}{n\alpha_n^{1/n}|E|^{(n-1)/n}} - 1,$$

and the Fraenkel asymmetry $\lambda(E)$ is

$$\lambda(E) = \min_{x\in\mathbb{R}^n} \frac{|E \setminus B^n(x,r)|}{|E|}.$$

The following sharp result, which was conjectured by Hall [9], gives an estimate for the asymmetry of a set in terms of the isoperimetric deficit.

Theorem 1.1. *(See Fusco, Maggi, and Pratelli [7].) Let E be Borel measurable, $0 < |E| < \infty$. Then*

$$\lambda(E) \leqslant C\delta(E)^{1/2},$$

where C depends only on n.

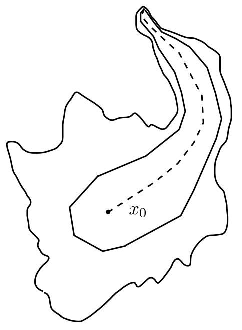

Fig. 2. John domain.

Figalli, Maggi, and Pratelli [3] showed that the constant C has polynomial growth in terms of n. Related problems around isoperimetric, Sobolev, and other geometric and functional inequalities are currently under active investigation, cf. [7,6,3], and the references therein.

In this paper, we take a different point of view concerning the extension of (1.2) to higher dimensions. Namely, we want to find the natural extensions of the sharp Bonnesen-style inequalities by restricting the class of domains in a suitable manner. For convex domains, the following extension was found by Fuglede [4]. The metric distortion $\beta(\Omega)$ of a bounded domain Ω is

$$\beta(\Omega) = \inf\left\{\frac{R-s}{s}\colon \text{there exists } x \text{ such that } B(x,s) \subset \Omega \subset B(x,R)\right\}.$$

Theorem 1.2. *(See Fuglede [4].) Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, be a bounded convex domain. Then*

$$\beta(\Omega) \leqslant \begin{cases} C(\delta(\Omega)\log\frac{1}{\delta(\Omega)})^{1/2}, & n=3, \\ C\delta(\Omega)^{2/(n+1)}, & n \geqslant 4. \end{cases}$$

The constants depend only on n and are explicitly calculable.

Theorem 1.2 gives a Bonnesen inequality in all dimensions for the class of convex domains. Fuglede also gives examples to show that the theorem is sharp in all dimensions, except for the constants. On the other hand, the counterexample we mentioned above hints that such inequalities might hold in much greater generality, namely for domains for which cusps do not occur. The main purpose of this paper is to prove sharp estimates which show that this is indeed the case. We will apply the familiar John domain condition, which in particular rules out outward cusps. (See Fig. 2.)

Definition 1.3. A bounded domain $\Omega \subset \mathbb{R}^n$ with a distinguished point $x_0 \in \Omega$ is called a John domain if there exists a constant $b > 0$ such that for all $x \in \Omega$ there is a path $\gamma : [0,\ell] \to \Omega$, parametrized by arclength, such that $\gamma(0) = x$, $\gamma(\ell) = x_0$, and

$$\operatorname{dist}\bigl(\gamma(t), \mathbb{R}^n \setminus \Omega\bigr) \geqslant bt \quad \text{for every } 0 \leqslant t \leqslant \ell.$$

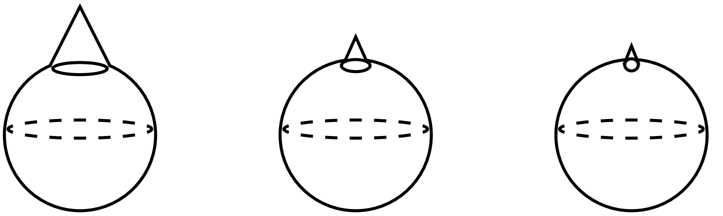

Fig. 3. Sharpness of Theorems 1.4 and 1.5, $n \geqslant 4$.

We will also use the notation (Ω, x_0) if we want to emphasize x_0. John domains naturally occur in connection with several areas of mathematics, such as conformal and quasiconformal geometry, and the theory of Sobolev functions.

Our first main theorem gives a sharp estimate for the outer metric distortion of a John domain. Let Ω be a bounded domain with circumradius R and volume radius r. We call $\alpha(\Omega) = (R - r)/r$ the outer metric distortion of Ω.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, be a John domain. Then*

$$\alpha(\Omega) \leqslant \varphi\big(\delta(\Omega)\big) := \begin{cases} C(\delta(\Omega)\log\frac{1}{\delta(\Omega)})^{1/2}, & n = 3, \\ C\delta(\Omega)^{1/(n-1)}, & n \geqslant 4. \end{cases} \tag{1.3}$$

The constants depend only on n and the constant b in Definition 1.3.

In order to achieve a Bonnesen inequality corresponding to (1.2) and Theorem 1.2, we also need to exclude inward cusps. Let (Ω, x_0) be a John domain. We denote

$$\Omega_0^c = B\big(x_0, 2\operatorname{diam}(\Omega)\big) \setminus \Omega.$$

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, be a John domain such that also Ω_0^c is a John domain. Moreover, assume that $\partial\Omega = \partial\Omega_0^c \cap B(x_0, 2\operatorname{diam}(\Omega))$. Then the estimate (1.3) holds with α replaced by β.*

Let us discuss the sharpness of Theorems 1.4 and 1.5. First, the decay rate in (1.3) is optimal. In dimension three this follows from the examples of Fuglede mentioned above. Theorem 1.5 actually extends Fuglede's results on convex and "nearly spherical" domains to a surprisingly large class of domains in dimension three (it is not difficult to see that the nearly spherical domains are John domains with quantitative bounds). For the higher dimensional case, optimality is seen as follows (see Fig. 3): fix $0 < \delta < \pi/10$, and let

$$\mathcal{C} = \big\{x \in \mathbb{R}^n\colon -x_n > \delta|x|,\ |x| < 1/10\big\}$$

be a truncated cone. Consider

$$\Omega_j = B^n \cup \big(\mathcal{C} + \big(1 + j^{-1}\big)e_n\big).$$

Then Ω_j satisfies the assumptions of Theorem 1.5 with John domain constant independent of j, $\delta(\Omega_j) \to 0$ as $j \to \infty$, and

$$C\beta(\Omega_j) \geqslant \delta(\Omega_j)^{1/(n-1)}$$

with C independent of j.

Second, our method does not give the best constant C in (1.3). However, thanks to the results of Figalli, Maggi, and Pratelli, and Fuglede mentioned above, with computable constants, our constant is also computable. We do not give an explicit form of this constant.

Fusco, Gelli, and Pisante [5] have independently proved a result similar to Theorems 1.4 and 1.5 for sets E, assuming that every boundary point of E satisfies a uniform cone condition with aperture $\pi/2$. Under such assumption, they control the Hausdorff distance of E and the closest ball by the right hand side of (1.3).

A particular class of domains to which Theorem 1.5 can be applied is provided by quasiconformal mappings. Namely, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a K-quasiconformal mapping. Then fB^n satisfies the assumptions of Theorem 1.5, with John domain constant depending only on K and n. Results similar to Theorem 1.5 in connection with quasiconformal analysis have been obtained in [13] and [14].

The proof of our main result involves an inequality of functions of one variable, Lemma 3.2. This inequality gives a quantitative version of the isoperimetric inequality for domains symmetric with respect to a line, see Theorem 3.1. The proof of this inequality is elementary, involving only divergence theorem and elementary algebraic inequalities. It gives, to the best of our knowledge, a new proof of the isoperimetric inequality. The main advantage is that it gives an explicit estimate for the isoperimetric deficit δ. We do not know if any other existing approach, such as the mass transportation approach, see e.g. [3] and [10], can be applied to obtain such sharp estimates for the outer distortion. We believe that our method is of independent interest.

Another step is to reduce the case of general domains to the 1-dimensional case above. This is done by using symmetrization methods. In this step Theorem 1.1 of Fusco, Maggi, and Pratelli, is applied, as well as Fuglede's Theorem 1.2.

2. Preliminaries

We denote the k-balls with center x and radius r by $B^k(x,r)$. Also, $B^k(0,r) = B^k(r)$, and $B^k(1) = B^k$. We use the notation $\alpha_k = |B^k|$, and $\omega_{k-1} = k\alpha_k$ is the $(k-1)$-measure of the boundary S^{k-1} of $B^k \subset \mathbb{R}^k$. The k-dimensional Hausdorff measure $\mathcal{H}^k$ is normalized such that $\mathcal{H}^k(B^k) = \alpha_k$.

Let $E \subset \mathbb{R}^n$ be Borel measurable, and $\Omega \subset \mathbb{R}^n$ a domain. The perimeter of E in Ω, $P(E,\Omega)$, is

$$P(E,\Omega) = \sup\left\{ \int_E \operatorname{div}\varphi\, dx \colon \varphi \in \left[C_c^1(\Omega)\right]^n,\ \|\varphi\|_\infty \leqslant 1 \right\}.$$

For smooth sets E, $P(E,\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)$. As usual, we denote $P(E,\mathbb{R}^n)$ by $P(E)$. See [1] for other basic properties.

Recall that the Schwarz symmetrization F of a Borel set E with respect to a line, say the nth coordinate axis, is defined as follows:

$$F \cap \{x_n = t\} = B^{n-1}(r_t),$$

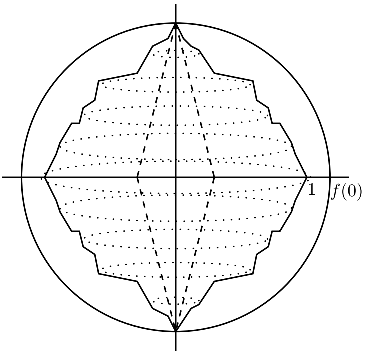

Fig. 4. Ω_f.

where we identify $\mathbb{R}^{n-1}$ with $\{x_n = t\}$, and r_t is defined by

$$\alpha_{n-1} r_t^{n-1} = \mathcal{H}^{n-1}\big(E \cap \{x_n = t\}\big).$$

The Steiner symmetrization G of E with respect to a hyperplane, say $\{x_n = t\}$, is defined as follows: given $s \in \mathbb{R}^{n-1}$, $G \cap \{x = (s, x_n)\}$ is the open line segment centered at t and with the same 1-measure as $E \cap \{x = (s, x_n)\}$. Both Schwarz and Steiner symmetrizations preserve the n-measure of E, and do not increase the perimeter, cf. [2].

3. Isoperimetric deficit for domains symmetric with respect to a line

Let $f : [0, 1] \to \mathbb{R}$ be a non-increasing and non-negative function such that $f(1) = 0$, $f(0) \geqslant 1$, and

$$f(t) \leqslant \sqrt{f(0)^2 - t^2} \quad \text{for all } t \in [0, 1]. \tag{3.1}$$

Moreover, we assume that

$$f(0) - f(t) \leqslant Mt \quad \text{for every } t \in [0, 1], \tag{3.2}$$

for some $M > 0$. We denote by Ω_f the interior of

$$\big\{x\colon |x_n| < f\big(|\tilde{x}|\big),\ |\tilde{x}| < 1\big\}, \tag{3.3}$$

where $x = (\tilde{x}, x_n) \in \mathbb{R}^n$ and $\tilde{x} = (x_1, \ldots, x_{n-1})$. Notice that (3.1) implies that $f(0)$ is the circumradius of Ω_f (see Fig. 4).

In this section, we prove that Theorem 1.4 holds for such a domain Ω_f.

Theorem 3.1. *Let f satisfy the above assumptions. Then the estimate (1.3) holds for Ω_f, with constants depending only on M and n.*

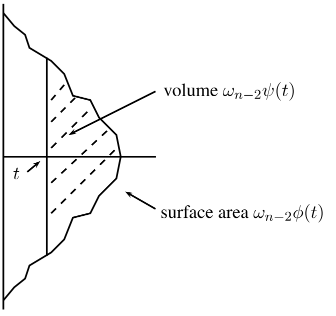

Fig. 5. ϕ and ψ.

We first notice that assumption (3.2) implies that Ω_f contains a ball of radius $\epsilon f(0)$, where ϵ depends only on M. Therefore, $\alpha(\Omega_f)$ is bounded from above by a constant depending only on M. This implies that in the proof of Theorem 3.1 we may assume that $\delta(\Omega_f) < 1$.

We may assume that $f \in C^\infty(0,1)$. Indeed, by applying [1, (3.47)] and cylindrical coordinates to our surface of revolution $\partial\Omega_f$, it suffices to show that f can be approximated in BV by smooth functions satisfying the above hypotheses. We extend f as $f(0)$ to $(-\infty,0)$ and as 0 to $(1,\infty)$, and take standard convolution approximations $g_\epsilon \in C^\infty$. Then $g_\epsilon(-\epsilon) = f(0)$ and $g_\epsilon(1+\epsilon) = 0$. We define $f_\epsilon(t) = g_\epsilon(-\epsilon + (1+2\epsilon)t)$. Then f_ϵ satisfies (3.1) and (3.2) with $2M$ when ϵ is small enough. So we may replace f by f_ϵ.

We use the following auxiliary functions:

$$\phi(t) = \int_t^1 \left(1 + f'(s)^2\right)^{\frac{1}{2}} s^{n-2}\, ds, \qquad \psi(t) = \int_t^1 f(s) s^{n-2}\, ds \tag{3.4}$$

for $t \in [0,1]$. The function ϕ represents the surface area of the section domain with parameter t up to a constant, and ψ the volume (see Fig. 5). Then

$$|\Omega_f| = 2\omega_{n-2}\psi(0), \qquad P(\Omega_f) = 2\omega_{n-2}\phi(0).$$

In Lemma 3.2 below, as well as in the proof, all of the inequalities become equalities when Ω_f is the unit ball, that is, when $f(t) = f_0(t) := \sqrt{1-t^2}$. In this case, we denote

$$\phi_0(t) = \int_t^1 \frac{s^{n-2}}{\sqrt{1-s^2}}\, ds, \qquad \psi_0(t) = \int_t^1 \sqrt{1-s^2}\, s^{n-2}\, ds. \tag{3.5}$$

Thus, we have

$$\alpha_n = \left|B^n\right| = 2\omega_{n-2}\psi_0(0), \qquad P\left(B^n\right) = 2\omega_{n-2}\phi_0(0).$$

Moreover, we also have

$$n\psi_0(0) = \phi_0(0). \tag{3.6}$$

We denote by ϕ_0^{-1} the inverse function of ϕ_0. We use also the following auxiliary functions in the proof of Theorem 3.1. We define $h : [0, \phi(0)] \to [0, \phi(0)/\phi_0(0)]$ as

$$h(t) = \frac{\phi(0)}{\phi_0(0)}\left(1 - \left[\phi_0^{-1}\left(\frac{\phi_0(0)}{\phi(0)}t\right)\right]^2\right)^{\frac{1}{2}}, \tag{3.7}$$

and define $g : [0, \phi(0)) \to [0, \infty)$ as

$$g(t) = \left[\phi_0^{-1}\left(\frac{\phi_0(0)}{\phi(0)}t\right)\right]^{3-n}.$$

In the case $n = 3$, the function g is the identity function and $h(t) = t$. It is easy to check that the function h is the prime function of g, that is,

$$h'(t) = g(t), \quad t \in \big(0, \phi(0)\big). \tag{3.8}$$

Now, we go back to Theorem 3.1. The isoperimetric deficit of Ω_f is

$$\delta(\Omega_f) = \frac{P(\Omega_f)}{n\alpha_n^{\frac{1}{n}}|\Omega_f|^{\frac{n-1}{n}}} - 1 = \frac{\psi_0(0)^{\frac{n-1}{n}}}{\phi_0(0)}\frac{\phi(0)}{\psi(0)^{\frac{n-1}{n}}} - 1. \tag{3.9}$$

The circumradius of Ω_f is $f(0)$, and the volume radius is $(2\omega_{n-2}\psi(0)/\alpha_n)^{\frac{1}{n}} = (\psi(0)/\psi_0(0))^{\frac{1}{n}}$. Therefore,

$$\alpha(\Omega_f) = \frac{\psi_0(0)^{\frac{1}{n}}}{\psi(0)^{\frac{1}{n}}} f(0) - 1. \tag{3.10}$$

Theorem 3.1 claims that

$$\alpha(\Omega_f) \leqslant \begin{cases} C(\delta(\Omega_f)\log\frac{1}{\delta(\Omega_f)})^{\frac{1}{2}}, & n = 3, \\ C\delta(\Omega_f)^{\frac{1}{n-1}}, & n \geqslant 4, \end{cases} \tag{3.11}$$

where $C = C(n, M) > 0$.

We will prove (3.11) in this section. The crux of the proof is the inequality in the following lemma. In the lemma, the functions ψ, ϕ and ψ_0, ϕ_0 are defined as in (3.4) and (3.5), respectively.

Lemma 3.2. *Let f be as in Theorem 3.1. We have*

$$\psi(0) \leqslant \psi_0(0)a^n - c(n)a(F + G), \tag{3.12}$$

where $c(n) > 0$, $a = (\phi(0)/\phi_0(0))^{\frac{1}{n-1}}$,

$$F = \int_0^1 \left(-f'(t) - \frac{1}{a}\big(1 + f'(t)^2\big)^{\frac{1}{2}}t\right)^2 \frac{t^{n-2}}{(1 + f'(t)^2)^{\frac{1}{2}}}\,dt, \tag{3.13}$$

$G = 0$ *when* $n = 3$; *and*

$$G = \int_0^1 \left(a^{\frac{1-n}{2}} g\big(\phi(t)\big)^{\frac{1}{2}} t^{\frac{n-1}{2}} - g\big(\phi(t)\big)^{-\frac{1}{n-3}}\right)^2 \big(1 + f'(t)^2\big)^{\frac{1}{2}} t^{n-2}\, dt$$

when $n \geqslant 4$.

Proof. To prove (3.12), we start with the left hand of the equality. Integration by parts gives us

$$\psi(0) = \int_0^1 f(t) t^{n-2}\, dt = -\frac{1}{n-1} \int_0^1 f'(t) t^{n-1}\, dt.$$

Then we use the trivial equality

$$2xy = x^2 + y^2 - (x-y)^2, \quad x, y \in \mathbb{R},$$

to write

$$\begin{aligned} -2f'(t)t^{n-1} = {} & \frac{1}{a}\big(1 + f'(t)^2\big)^{\frac{1}{2}} t^n + a\frac{f'(t)^2 t^{n-2}}{(1 + f'(t)^2)^{\frac{1}{2}}} \\ & - a\left(-f'(t) - \frac{1}{a}\big(1 + f'(t)^2\big)^{\frac{1}{2}} t\right)^2 \frac{t^{n-2}}{(1 + f'(t)^2)^{\frac{1}{2}}}. \end{aligned}$$

Here and in the following, the constant $a = (\phi(0)/\phi_0(0))^{\frac{1}{n-1}}$ is the same as in the lemma. Thus we arrive at

$$\begin{aligned} 2(n-1)\psi(0) = {} & \frac{1}{a} \int_0^1 \big(1 + f'(t)^2\big)^{\frac{1}{2}} t^n\, dt \\ & + a \int_0^1 \frac{f'(t)^2}{(1 + f'(t)^2)^{\frac{1}{2}}} t^{n-2}\, dt - aF, \end{aligned} \tag{3.14}$$

where F is as in (3.13). The remaining part of the proof is to estimate the first integral in the right hand side of (3.14). We claim that

$$\begin{aligned} \int_0^1 \big(1 + f'(t)^2\big)^{\frac{1}{2}} t^n\, dt \leqslant {} & a^2 \int_0^1 \frac{t^{n-2}}{(1 + f'(t)^2)^{\frac{1}{2}}}\, dt \\ & + (n-2)\psi_0(0)a^{n+1} - c(n)a^2 G, \end{aligned} \tag{3.15}$$

where G is defined as in the lemma. Combining (3.14) and (3.15), we obtain that

$$2(n-1)\psi(0) \leqslant a\phi(0) + (n-2)\psi_0(0)a^n - aF - c(n)aG,$$

from which, together with the fact (3.6), follows (3.12).

Therefore, it only remains to prove the claimed inequality (3.15). We need the following consequence of the Cauchy–Schwarz inequality:

$$t^{n-1} = (n-1)\int_0^t s^{n-2}\,ds \leqslant (n-1)J(t)^{\frac{1}{2}}H(t)^{\frac{1}{2}}, \tag{3.16}$$

where

$$J(t) = \int_0^t \frac{s^{n-2}}{h(\phi(s))(1+f'(s)^2)^{\frac{1}{2}}}\,ds, \tag{3.17}$$

and

$$H(t) = \int_0^t h\big(\phi(s)\big)\big(1+f'(s)^2\big)^{\frac{1}{2}}s^{n-2}\,ds = \int_0^t h\big(\phi(s)\big)\,d\big(-\phi(s)\big), \tag{3.18}$$

where the function h is defined as in (3.7) and ϕ is as in (3.4). Now we estimate the first integral in the right hand side of (3.14) as follows. We write

$$\int_0^1 \big(1+f'(t)^2\big)^{\frac{1}{2}}t^n\,dt = \int_0^1 t^2\,d\big(-\phi(t)\big). \tag{3.19}$$

Then we use the inequality

$$u^\gamma v^{1-\gamma} \leqslant \gamma u + (1-\gamma)v - c(\sqrt{u}-\sqrt{v})^2, \quad u, v \geqslant 0,$$

for all $\gamma \in (0,1)$, where $c = c(\gamma) > 0$. We let $\gamma = 2/(n-1)$ and

$$u = u(t) = a^{3-n}g\big(\phi(t)\big)t^{n-1}, \qquad v = v(t) = a^2 g\big(\phi(t)\big)^{-\frac{2}{n-3}}.$$

We obtain

$$t^2 = u(t)^{\frac{2}{n-1}}v(t)^{\frac{n-3}{n-1}} \leqslant \frac{2}{n-1}u(t) + \frac{n-3}{n-1}v(t) - c(n)\big(\sqrt{u(t)}-\sqrt{v(t)}\big)^2,$$

where $c(n) > 0$ is a constant when $n \geqslant 4$. We comment on the above inequality in the special case $n = 3$. In this case we have $\gamma = 1$. We let $c(n) = 0$. So, the last two terms in the above inequality vanish in this case. Recall that $g(s) = 1$ when $n = 3$. So the above inequality is trivial. Thus we have from (3.19) that

$$\int_0^1 \left(1+f'(t)^2\right)^{\frac{1}{2}} t^n\,dt \leqslant \frac{2}{n-1}\int_0^1 u(t)\,d\left(-\phi(t)\right) + \frac{n-3}{n-1}\int_0^1 v(t)\,d\left(-\phi(t)\right) - c(n)a^2 G, \tag{3.20}$$

where G is defined as in the lemma. We remark here that when $n=3$, this term vanishes. In this case, we let $G=0$.

To finish the proof of (3.15), we only need to estimate the two integrals on the right side of (3.20). For the second one, we have

$$\int_0^1 v(t)\,d\left(-\phi(t)\right) = a^{n+1}(n-1)\psi_0(0), \tag{3.21}$$

where ψ_0 is defined as in (3.5). Indeed, by changing of variables, and integrating by parts,

$$\begin{aligned}
\int_0^1 v(t)\,d\left(-\phi(t)\right) &= a^2\int_0^1 g\left(\phi(t)\right)^{-\frac{2}{n-3}}\,d\left(-\phi(t)\right)\\
&= a^2\int_0^{\phi(0)}\left[\phi_0^{-1}\left(\frac{\phi_0(0)}{\phi(0)}s\right)\right]^2 ds\\
&= a^2\frac{\phi(0)}{\phi_0(0)}\int_0^1 \frac{r^n}{\sqrt{1-r^2}}\,dr = a^{n+1}(n-1)\psi_0(0).
\end{aligned}$$

This proves (3.21). Then we apply (3.16) to estimate the first integral on the right side of (3.20). We have

$$\begin{aligned}
&\int_0^1 u(t)\,d\left(-\phi(t)\right)\\
&= a^{3-n}\int_0^1 g\left(\phi(t)\right)t^{n-1}\,d\left(-\phi(t)\right) = a^{3-n}\int_0^1 t^{n-1}\,d\left(-h\left(\phi(t)\right)\right)\\
&\leqslant a^{3-n}(n-1)\int_0^1 J(t)^{\frac{1}{2}}H(t)^{\frac{1}{2}}\,d\left(-h\left(\phi(t)\right)\right)\\
&\leqslant a^{3-n}(n-1)\left(\int_0^1 J(t)\,d\left(-h\left(\phi(t)\right)\right)\right)^{\frac{1}{2}}\left(\int_0^1 H(t)\,d\left(-h\left(\phi(t)\right)\right)\right)^{\frac{1}{2}},
\end{aligned} \tag{3.22}$$

where the second equality follows from the fact (3.8) that the function h is the prime function of g, and the last inequality from the Cauchy–Schwarz inequality. Recall that the functions J and H are defined in (3.17) and (3.18), respectively. We observe that by integration by parts,

$$\int_0^1 J(t)\,d\bigl(-h\bigl(\phi(t)\bigr)\bigr) = \int_0^1 \frac{t^{n-2}}{(1+f'(t)^2)^{\frac{1}{2}}}\,dt. \tag{3.23}$$

We claim that

$$\int_0^1 H(t)\,d\bigl(-h\bigl(\phi(t)\bigr)\bigr) = a^{3n-3}\psi_0(0). \tag{3.24}$$

Indeed, by changing of variables and integration by parts, we have

$$\begin{aligned}
\int_0^1 H(t)\,d\bigl(-h\bigl(\phi(t)\bigr)\bigr) &= \int_0^1\int_0^t h\bigl(\phi(s)\bigr)\,d\bigl(-\phi(s)\bigr)\,d\bigl(-h\bigl(\phi(t)\bigr)\bigr)\\
&= \int_0^{\phi(0)}\int_w^{\phi(0)} h(r)\,dr\,d\bigl(h(w)\bigr)\\
&= \int_0^{\phi(0)} h(w)^2\,dw\\
&= \left(\frac{\phi(0)}{\phi_0(0)}\right)^2 \int_0^{\phi(0)} 1-\left[\phi_0^{-1}\left(\frac{\phi_0(0)}{\phi(0)}w\right)\right]^2 dw\\
&= \left(\frac{\phi(0)}{\phi_0(0)}\right)^3 \int_0^1 \sqrt{1-r^2}\,r^{n-2}\,dr\\
&= a^{3n-3}\psi_0(0),
\end{aligned}$$

which proves (3.24). Now we plug the equalities (3.23) and (3.24) into (3.22) and obtain

$$\int_0^1 u(t)\,d\bigl(-\phi(t)\bigr) \leqslant (n-1)\psi_0(0)^{\frac{1}{2}}a^{\frac{n+3}{2}}\left(\int_0^1 \frac{t^{n-2}}{(1+f'(t)^2)^{\frac{1}{2}}}\,dt\right)^{\frac{1}{2}}. \tag{3.25}$$

To conclude the proof of (3.15), we go back to (3.20). Plugging the estimates (3.25) and (3.21) into it, we obtain

$$\int_0^1 \left(1+f'(t)^2\right)^{\frac{1}{2}} t^n\,dt \leqslant 2\psi_0(0)^{\frac{1}{2}} a^{\frac{n+3}{2}} \left(\int_0^1 \frac{t^{n-2}}{(1+f'(t)^2)^{\frac{1}{2}}}\,dt\right)^{\frac{1}{2}}$$
$$+ (n-3)a^{n+1}\psi_0(0) - c(n)a^2 G,$$

from which (3.15) follows by the Cauchy–Schwarz inequality. This proves the claimed inequality (3.15), and hence the lemma. □

Now we can rewrite Lemma 3.2 to get an explicit estimate for the isoperimetric deficit $\delta(\Omega_f)$ from below.

Corollary 3.3. *Let f be as in Theorem 3.1. Suppose that $\delta(\Omega_f) \leqslant 1$. Then we have*

$$F + G \leqslant c(n)a^{n-1}\delta(\Omega_f), \tag{3.26}$$

where F, G and a are as in Lemma 3.2.

Proof. We start with the isoperimetric deficit $\delta(\Omega_f)$ and rewrite (3.9) as

$$1 + \delta(\Omega_f) = \left(\frac{\psi_0(0)}{\psi(0)} a^n\right)^{\frac{n-1}{n}}, \tag{3.27}$$

where $a = (\phi(0)/\phi_0(0))^{\frac{1}{n-1}}$ is the same as in Lemma 3.2. Then we rewrite (3.12) as follows:

$$F + G \leqslant c(n)\frac{\psi(0)}{a}\left(\frac{\psi_0(0)}{\psi(0)}a^n - 1\right)$$
$$\leqslant c(n)a^{n-1}\left(\left(1+\delta(\Omega_f)\right)^{\frac{n}{n-1}} - 1\right) \leqslant c(n)a^{n-1}\delta(\Omega_f),$$

since we assume that $\delta(\Omega_f) \leqslant 1$. This proves the corollary. □

In the remaining part of this section, we prove (3.11), and hence Theorem 3.1. We will show that the outer metric distortion $\alpha(\Omega_f)$ of the domain Ω_f can be controlled by the integrals F and G in Lemma 3.2, and therefore by the isoperimetric deficit $\delta(\Omega_f)$.

Proof of Theorem 3.1. We will estimate the integrals F and G in Lemma 3.2 from below. First, we deal with F. Let $\theta \in (0,1)$ be chosen later. By Hölder's inequality,

$$\left|\int_\theta^1 \left(-f'(t) - \frac{1}{a}\left(1+f'(t)^2\right)^{\frac{1}{2}} t\right) dt\right| \leqslant F^{\frac{1}{2}} \left(\int_\theta^1 \frac{(1+f'(t)^2)^{\frac{1}{2}}}{t^{n-2}}\,dt\right)^{\frac{1}{2}}.$$

We denote

$$\rho(\theta) = \int_\theta^1 \left(1+f'(t)^2\right)^{\frac{1}{2}} t\,dt.$$

The integral on the left side of the above inequality is $f(\theta)-\rho(\theta)/a$. Thus we obtain

$$\left|f(\theta)-\frac{\rho(\theta)}{a}\right| \leqslant F^{\frac{1}{2}}\left(\int_\theta^1 \frac{(1+f'(t)^2)^{\frac{1}{2}}}{t^{n-2}}\,dt\right)^{\frac{1}{2}}. \tag{3.28}$$

We continue to estimate the integral on the right side of the above inequality. It is easy to prove that

$$t \leqslant \varphi(t) := \int_0^t \bigl(1+f'(s)^2\bigr)^{\frac{1}{2}}\,ds \leqslant (M+1)t, \quad t\in[0,1]. \tag{3.29}$$

Indeed, the first inequality is trivial, and the second follows from the assumption (3.2) on the non-increasing function f, which we have not used before,

$$\varphi(t) \leqslant \int_0^t \bigl(1-f'(s)\bigr)\,ds = t+\bigl(f(0)-f(t)\bigr) \leqslant (M+1)t.$$

This proves (3.29). Here, we also estimate $\phi(0)$. We have

$$\frac{1}{n-1} \leqslant \phi(0) = \int_0^1 \bigl(1+f'(t)^2\bigr)^{\frac{1}{2}} t^{n-2}\,dt \leqslant \varphi(0) \leqslant M+1.$$

Therefore, we have the following estimate for $a=(\phi(0)/\phi_0(0))^{\frac{1}{n-1}}$

$$a \approx 1. \tag{3.30}$$

Here and in the following, we denote $A\approx B$ if two quantities A and B are comparable, that is, if there are constants $c_1>0, c_2>0$ depending only on n and M, such that $c_1 A \leqslant B \leqslant c_2 A$. Now we apply (3.29) to estimate the integral on the right side of (3.28):

$$\begin{aligned}\int_\theta^1 \frac{(1+f'(t)^2)^{\frac{1}{2}}}{t^{n-2}}\,dt &\leqslant (M+1)^{n-2}\int_\theta^1 \frac{(1+f'(t)^2)^{\frac{1}{2}}}{\varphi(t)^{n-2}}\,dt\\ &= (M+1)^{n-2}\int_\theta^1 \frac{1}{\varphi(t)^{n-2}}\,d\bigl(\varphi(t)\bigr),\end{aligned}$$

from which we deduce

$$\int_\theta^1 \frac{(1+f'(t)^2)^{\frac{1}{2}}}{t}\,dt \leqslant c(M)\log\frac{\varphi(1)}{\varphi(\theta)} \leqslant (M+1)\log\frac{M+1}{\theta},$$

and

$$\int_{\theta}^{1} \frac{(1+f'(t)^2)^{\frac{1}{2}}}{t^{n-2}}\,dt \leqslant c(n,M)\theta^{3-n}, \tag{3.31}$$

when $n \geqslant 4$. Now it follows from (3.28) that

$$\left| f(\theta) - \frac{\rho(\theta)}{a} \right| \leqslant cF^{\frac{1}{2}}\Phi(\theta) \leqslant c\delta(\Omega_f)^{\frac{1}{2}}\Phi(\theta), \tag{3.32}$$

where $c = c(n,M) > 0$ and

$$\Phi(\theta) = \begin{cases} \log^{\frac{1}{2}} \frac{M+1}{\theta}, & n = 3; \\ \theta^{\frac{3-n}{2}}, & n \geqslant 4. \end{cases}$$

Here we used the estimate (3.26) on F and the estimate (3.30) on a.

Second, we deal with the integral G. We will show that

$$\left|\rho(\theta) - a^2\right| \leqslant cG^{\frac{1}{2}}\Phi(\theta) + c\theta^2, \tag{3.33}$$

where $c = c(n,M) > 0$ and Φ is as above. We divide the proof of (3.33) into two cases: $n = 3$ and $n \geqslant 4$. In the case $n = 3$, we have $\rho(\theta) = \phi(\theta)$ and $a^2 = \phi(0)/\phi_0(0) = \phi(0)$. Thus we have by (3.29) that

$$\left|\rho(\theta) - a^2\right| = \left|\phi(\theta) - \phi(0)\right| = \int_0^{\theta} \left(1 + f'(t)^2\right)^{\frac{1}{2}} t\,dt \leqslant \theta\varphi(\theta) \leqslant c\theta^2,$$

which proves (3.33) in the case $n = 3$.

To prove (3.33) in the case $n \geqslant 4$, we start with the following estimate. It follows from (3.29) that

$$\frac{1}{n-1}t^{n-1} \leqslant \phi(0) - \phi(t) = \int_0^t \left(1 + f'(s)^2\right)^{\frac{1}{2}} s^{n-2}\,ds \leqslant t^{n-2}\varphi(t) \leqslant c(n,M)t^{n-1}.$$

The function ϕ_0^{-1} has the following property:

$$\phi_0^{-1}\left(\phi_0(0) - s\right) \approx s^{\frac{1}{n-1}}, \quad s \in \left[0, \phi_0(0)\right].$$

Thus, from the above two estimates, we deduce that

$$\phi_0^{-1}\left(\frac{\phi_0(0)}{\phi(0)}\phi(t)\right) \approx t, \quad t \in [0,1],$$

and therefore

$$g\left(\phi(t)\right) \approx t^{3-n}. \tag{3.34}$$

Now we go back to the integral G. We denote

$$\tilde{u}(t) = a^{\frac{1-n}{2}} g\big(\phi(t)\big)^{\frac{1}{2}} t^{\frac{n-1}{2}}, \qquad \tilde{v}(t) = g\big(\phi(t)\big)^{-\frac{1}{n-3}}.$$

Then

$$G = \int_0^1 \big|\tilde{u}(t) - \tilde{v}(t)\big|^2\, dt.$$

We note that it follows from (3.34) that

$$\tilde{u}(t) + \tilde{v}(t) \approx t, \quad t \in [0,1]. \tag{3.35}$$

Let $I(t) = a^{3-n} g(\phi(t)) - t^{3-n}$. We will show that

$$\big|I(t)\big| \leqslant ct^{2-n}\big|\tilde{u}(t) - \tilde{v}(t)\big|, \quad t \in [0,1]. \tag{3.36}$$

Indeed, we apply the elementary inequality

$$\big|x^\gamma - y^\gamma\big| \leqslant c(\gamma)|x-y|(x+y)^{\gamma-1}, \quad x, y \geqslant 0,\ \gamma > 0,\ c(\gamma) > 0$$

with $x = \tilde{u}(t)$, $y = \tilde{v}(t)$ and $\gamma = 2(n-3)/(n-1)$ to obtain

$$\begin{aligned}\Big|a^{3-n} g\big(\phi(t)\big)^{\frac{n-3}{n-1}} t^{n-3} - g\big(\phi(t)\big)^{-\frac{2}{n-1}}\Big| &\leqslant c\big|\tilde{u}(t) - \tilde{v}(t)\big|\big(\tilde{u}(t) + \tilde{v}(t)\big)^{\frac{n-5}{n-1}} \\ &\leqslant ct^{\frac{n-5}{n-1}}\big|\tilde{u}(t) - \tilde{v}(t)\big|,\end{aligned}$$

where the last step follows from (3.35). The left side of the above inequality is $t^{n-3} g(\phi(t))^{-\frac{2}{n-1}} \times |I(t)|$. Thus

$$\big|I(t)\big| \leqslant ct^{3-n} t^{\frac{n-5}{n-1}} g\big(\phi(t)\big)^{\frac{2}{n-1}}\big|\tilde{u}(t) - \tilde{v}(t)\big|,$$

from which, together with (3.34), the claimed inequality (3.36) follows. Now by (3.36) and the Cauchy–Schwarz inequality, we have

$$\left|\int_\theta^1 I(t)\, d\big(-\phi(t)\big)\right| \leqslant G^{\frac{1}{2}} \left(\int_\theta^1 \frac{(1 + f'(t)^2)^{\frac{1}{2}}}{t^{n-2}}\, dt\right)^{\frac{1}{2}} \leqslant cG^{\frac{1}{2}} \Phi(\theta), \tag{3.37}$$

where $c = c(n, M) > 0$. The last inequality follows from (3.31). We estimate the integral of I as follows. We note that

$$\int_\theta^1 I(t)\, d\big(-\phi(t)\big) = a^{3-n} \int_\theta^1 g\big(\phi(t)\big)\, d\big(-\phi(t)\big) - \rho(\theta).$$

We have by changing variables that

$$\int_0^1 g(\phi(t))\,d(-\phi(t)) = \frac{\phi(0)}{\phi_0(0)} \int_0^1 \frac{s}{\sqrt{1-s^2}}\,ds = \frac{\phi(0)}{\phi_0(0)} = a^{n-1}.$$

We also have by (3.34) and (3.29) that

$$0 \leqslant \int_0^\theta g(\phi(t))\,d(-\phi(t)) \leqslant c \int_0^\theta \left(1+f'(t)^2\right)^{\frac{1}{2}} t\,dt \leqslant c\theta\varphi(\theta) \leqslant c\theta^2.$$

It then follows from the above three inequalities that

$$\left| \int_\theta^1 I(t)\,d(-\phi(t)) \right| \geqslant \left|\rho(\theta) - a^2\right| - c\theta^2,$$

from which, together with (3.37), follows the claimed inequality (3.33).

Finally, we combine (3.32) and (3.33) to obtain

$$\left|f(\theta) - a\right| \leqslant c\delta(\Omega_f)^{\frac{1}{2}}\Phi(\theta) + c\theta^2.$$

We assume (3.2) on f, that is, $0 \leqslant f(0) - f(\theta) \leqslant M\theta$, $\theta \in [0,1]$. Thus we have

$$\left|f(0) - a\right| \leqslant c\delta(\Omega_f)^{\frac{1}{2}}\Phi(\theta) + c\theta. \tag{3.38}$$

Now we estimate the outer metric distortion $\alpha(\Omega_f)$. By (3.10) and (3.27),

$$\alpha(\Omega_f) = \left(1+\delta(\Omega_f)\right)^{\frac{1}{n-1}} \frac{f(0)}{a} - 1 \leqslant c\left|f(0) - a\right| + c\delta(\Omega_f).$$

Thus, by (3.38), we obtain the following estimate for $\alpha(\Omega_f)$

$$\alpha(\Omega_f) \leqslant c\delta(\Omega_f)^{\frac{1}{2}}\Phi(\theta) + c\theta + c\delta(\Omega_f),$$

which holds for all $\theta \in (0,1)$. Then we choose $\theta = \delta(\Omega_f)^{\frac{1}{n-1}}$ and we obtain the desired estimate (3.11). The proof of Theorem 3.1 is finished. □

4. Proof of Theorem 1.4

Let $\Omega = (\Omega, x_0)$ be a John domain with constant b. We now prove Theorem 1.4. We first assume that $\delta(\Omega) \geqslant \delta_0$, where $\delta_0 > 0$ is a small number to be determined later. The John domain condition now implies that there exists a ball $B(x, \epsilon\,\mathrm{diam}(\Omega)) \subset \Omega$, where $\epsilon > 0$ depends only on b. Since the circumradius R is bounded by $\operatorname{diam}(\Omega)$, we have

$$\alpha(\Omega) \leqslant \frac{\operatorname{diam}\Omega - r}{r} \leqslant \epsilon^{-1} \leqslant C(\epsilon, \delta_0)\varphi\big(\delta(\Omega)\big).$$

Thus (1.3) holds when $\delta(\Omega) \geqslant \delta_0$. The case $\delta(\Omega) \leqslant \delta_0$ is proved by combining Proposition 4.1 below, where the choice of δ_0 is made, with Theorem 3.1.

Proposition 4.1. *Let $(\Omega, x_0) \subset \mathbb{R}^n$ be a John domain. Then there exists $\delta_0 > 0$, such that if $\delta(\Omega) \leqslant \delta_0$, then the following holds: there exists $f : [0,1] \to [0, f(0)]$ such that f satisfies the assumptions of Theorem 3.1, and such that Ω_f satisfies $\delta(\Omega_f) \leqslant 2\delta(\Omega)$ and $2\alpha(\Omega_f) \geqslant \alpha(\Omega)$. Here δ_0 and constant M in (3.2) depend only on n and the John domain constant b.*

Proof. Let $B^n(x,r)$ be a ball which realizes the asymmetry $\lambda(\Omega)$. By translating Ω, if necessary, we may assume that $x = 0$. Let u be the smallest radius such that $\Omega \subset B^n(u)$. Then the circumradius of Ω is not larger than u. Fix a point

$$a \in \partial\Omega \cap S^{n-1}(u).$$

By rotating Ω about the origin, if necessary, we may assume that $a = ue_n$.

Recall that $\lambda(\Omega) \leqslant C(n)\delta(\Omega)^{1/2}$ by Theorem 1.1. In particular, since $\lambda(\Omega)$ is realized by $B^n(r)$, we conclude that if $\delta(\Omega)$ is small enough depending on n, then there exists

$$v \in \Omega \cap B^n(r) \cap \{x_n < -3r/4\}.$$

If $(x_0)_n > -r/2$, we can apply the John domain condition to the point v to conclude that for every $-r/2 < \mu < (x_0)_n$, there exists a point $y(\mu) \in \Omega$ such that $(y(\mu))_n = \mu$ and

$$\operatorname{dist}\big(y(\mu), \mathbb{R}^n \setminus \Omega\big) \geqslant Cbr. \tag{4.1}$$

We can also apply the John domain condition to points converging to ue_n to see that for every $(x_0)_n \leqslant \mu < u$ there exists a point $y(\mu) \in \Omega$ such that $(y(\mu))_n = \mu$ and

$$\operatorname{dist}\big(y(\mu), \mathbb{R}^n \setminus \Omega\big) \geqslant b\big|ue_n - y(\mu)\big|. \tag{4.2}$$

Combining (4.1) and (4.2), we conclude that (independent of the position of x_0), there exists a constant $C > 0$, only depending on b, such that

$$B\big(y(\mu), C\big(u - y(\mu)_n\big)\big) \subset \Omega \tag{4.3}$$

for every $-r/2 < \mu < u$. From now on we use the John domain property of Ω only when we refer to (4.3).

We now perform Schwarz symmetrization with respect to the x_n-axis. We thus obtain Ω' such that $\Omega' \cap \{x_n = t\}$ is an $(n-1)$-ball with center $(0, 0, \ldots, t)$. In particular,

$$\alpha_n r^n \lambda(\Omega) = \big|B^n(r) \setminus \Omega\big| \geqslant \big|B^n(r) \setminus \Omega'\big|.$$

Moreover, $|\Omega'| = |\Omega|$, $\delta(\Omega') \leqslant \delta(\Omega)$, and ue_n belongs to the boundary of Ω'.

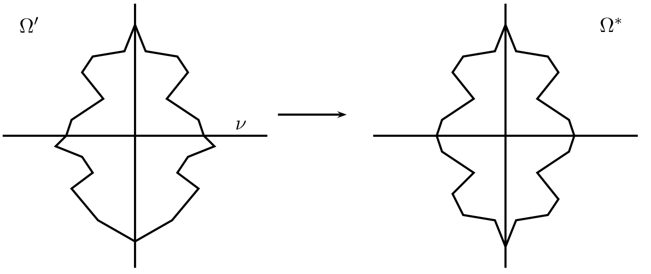

Fig. 6. Reflection.

Next, let $\nu \in \mathbb{R}$ satisfy $|\Omega_\nu^+| = |\Omega' \setminus \Omega_\nu^+| = \alpha_n r^n/2$, where

$$\Omega_\nu^+ = \Omega' \cap \{x_n \geqslant \nu\}.$$

Then, if $\nu > 0$, we have

$$\left|B^n(r) \cap \{x_n < \nu\}\right| \geqslant \alpha_n r^n/2 + C \min\{\nu, r\} r^{n-1}.$$

Therefore,

$$\begin{aligned}\alpha_n r^n \lambda(\Omega) &\geqslant \left|B^n(r) \setminus \Omega'\right| \geqslant \left|B^n(r) \cap \{x_n < \nu\} \setminus \Omega'\right| \\ &\geqslant \left|B^n(r)\right|/2 + C \min\{\nu, r\} r^{n-1} - \left|B^n(r)\right|/2 \geqslant C \min\{\nu, r\} r^{n-1}\end{aligned}$$

because $|\Omega' \setminus \Omega_\nu^+| = |B^n(r)|/2$. Using Theorem 1.1, we conclude that

$$\frac{\min\{\nu, r\}}{r} \leqslant C\lambda(\Omega) \leqslant C\delta(\Omega)^{1/2} \leqslant C\varphi\big(\delta(\Omega)\big)$$

when $\delta(\Omega)$ is small enough. Notice that it also follows that $\nu < r/2$ when $\delta(\Omega)$ is small enough. Similarly, we conclude that $\nu \geqslant -r/2$ when $\delta(\Omega)$ is small enough. Therefore, we can choose δ_0, only depending on n, such that

$$-r/2 < \nu < \frac{u-r}{2} \quad \text{when } \delta(\Omega) \leqslant \delta_0. \tag{4.4}$$

Notice in particular that (4.3) applies for all $\nu < \mu < u$.

We define

$$\Omega^* = \Omega_\nu^+ \cup T\big(\Omega_\nu^+\big),$$

where T is the reflection with respect to $\{x_n = \nu\}$. Then $|\Omega^*| = |\Omega| = |B^n(r)|$, and the closure of Ω^* contains both $a = ue_n$ and $T(a) = (2\nu - u)e_n$. (See Fig. 6.)

We next estimate the deficit $\delta(\Omega^*)$. By the relative isoperimetric inequality,

$$n\alpha_n^{1/n}\left|\Omega' \setminus \Omega_\nu^+\right|^{(n-1)/n} \leqslant \omega_{n-1} r^{n-1}/2.$$

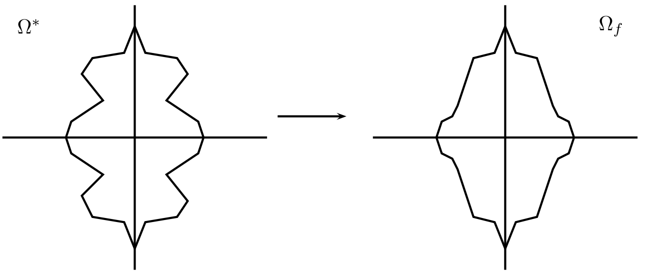

Fig. 7. Steiner symmetrization.

We have

$$P(\Omega') \geqslant \omega_{n-1} r^{n-1}/2 + P(\Omega_\nu^+, \{x_n > \nu\})$$
$$\geqslant n\alpha_n^{1/n} |\Omega|^{(n-1)/n}/2 + P(\Omega^*)/2.$$

Therefore, since $|\Omega^*| = |\Omega'| = |\Omega|$,

$$P(\Omega^*) - n\alpha_n^{1/n} |\Omega'|^{(n-1)/n} \leqslant 2\big(P(\Omega') - n\alpha_n^{1/n} |\Omega|^{(n-1)/n}\big).$$

We conclude that $\delta(\Omega^*) \leqslant 2\delta(\Omega') \leqslant 2\delta(\Omega)$.

We next estimate $\alpha(\Omega^*)$. By (4.4), the circumradius of Ω^* is bounded from below by

$$\big|a - T(a)\big|/2 \geqslant u - (u-r)/2 = (u-r)/2 + r.$$

But the circumradius of Ω is at most u, so $2\alpha(\Omega^*) \geqslant \alpha(\Omega)$ when δ_0 is small enough. Also, we notice that (4.3) and the definition of Ω^* imply

$$\mathcal{H}^{n-1}\big(\Omega^* \cap \{x_n = t\}\big) \geqslant C(u-t)^{n-1} \tag{4.5}$$

when $t \geqslant \nu$. By symmetry, similar estimate holds when $t < \nu$.

Finally, we perform another symmetrization, this time a Steiner symmetrization with respect to the hyperplane $\{x_n = \nu\}$ (see Fig. 7). We translate such that $\nu = 0$. Then the resulting domain is of the form Ω_f as in (3.3). Also, $\alpha(\Omega_f) = \alpha(\Omega^*)$ by construction. By (4.5),

$$f(|\tilde{x}|) \geqslant \mathcal{H}^1\big(\{u - t > C^{-1}|\tilde{x}|\}\big) = \mathcal{H}^1\big(\{t < u - C^{-1}|\tilde{x}|\}\big) = u - C^{-1}|\tilde{x}|,$$

and so

$$f(0) - f(|\tilde{x}|) = u - f(|\tilde{x}|) \leqslant C^{-1}|\tilde{x}|.$$

Also, f is clearly non-increasing and satisfies $f(1) = 0$ and $f(|\tilde{x}|) \leqslant \sqrt{f(0)^2 - |\tilde{x}|^2}$ if we scale Ω_f as follows: if $f(b) = 0$, let $g(a) = b^{-1} f(ba)$. Then replace Ω_f by Ω_g if necessary. The proof is complete. □

5. Proof of Theorem 1.5

In this section we assume that Ω satisfies the assumptions of Theorem 1.5. Also, the constants appearing will depend only on n and the John domain datas of Ω and Ω_0^c. We first notice that the argument in the beginning of Section 4 shows that we can assume $\delta(\Omega) \leqslant \delta_0$, where δ_0 can be chosen to be as small as desired.

Let r be the volume radius of Ω. By translating Ω if necessary, we may assume that $B^n(r)$ realizes the Fraenkel asymmetry of Ω. Let s and t be the largest and smallest radii, respectively, such that

$$B^n(s) \subset \Omega \subset B^n(t).$$

Then

$$\beta(\Omega) \leqslant \frac{t-s}{r},$$

so it suffices to estimate $(t-s)/r$. We have the following nonsharp version of Theorem 1.5.

Lemma 5.1. *If δ_0 is small enough, then*

$$\frac{t-s}{r} \leqslant C\lambda(\Omega)^{1/n} \leqslant C\delta(\Omega)^{1/2n}.$$

Proof. Using the John domain property of Ω, and the fact $S^{n-1}(t) \cap \partial\Omega \neq \emptyset$, we find a ball

$$B^n\big(x, C(t-r)\big) \subset \Omega \setminus B^n(r).$$

We conclude that

$$C(t-r)^n \leqslant \lambda(\Omega)r^n. \tag{5.1}$$

Similarly, since $S^{n-1}(s) \setminus \Omega \neq \emptyset$ ($0 \in \Omega_0^c$ if $s = 0$), the John domain property of Ω_0^c gives a ball

$$B^n\big(y, C(r-s)\big) \subset \Omega_0^c \cap B^n(r) = B^n(r) \setminus \Omega.$$

We conclude that

$$C(r-s)^n \leqslant \lambda(\Omega)r^n. \tag{5.2}$$

The first inequality follows by combining (5.1) and (5.2), and the second by Theorem 1.1. □

Lemma 5.1 implies that if $\delta(\Omega)$ is assumed to be small enough, then s is close to r, and strictly positive in particular. We can estimate t using Theorem 1.4.

Lemma 5.2. *We have*

$$\frac{t-s}{r} \leqslant 2\varphi\big(\delta(\Omega)\big) + 2\frac{r-s}{r}.$$

Proof. Let R be the circumradius of Ω, and $\Omega \subset B^n(x, R)$. Since

$$B^n(s) \subset \Omega \subset B^n(x, R),$$

we have $|x| < R - s$. We conclude that

$$\Omega \subset B^n(x, R) \subset B^n\big(R + (R - s)\big).$$

In particular,

$$t - s \leqslant R + (R - s) - s = 2(R - r) + 2(r - s).$$

But $R - r$ can be estimated by $r\varphi(\delta(\Omega))$ using Theorem 1.4. The lemma follows. □

In view of Lemma 5.2, Theorem 1.5 follows if we can estimate $r - s$ properly. We start by defining functions g and h as follows:

$$g(u) = \mathcal{H}^{n-1}\big(S^{n-1}(u) \setminus \Omega\big) \quad \text{and}$$
$$h(u) = \mathcal{H}^{n-1}\big(S^{n-1}(u) \cap \Omega\big) = \omega_{n-1}u^{n-1} - g(u).$$

Notice that $g(u) = 0$ when $u < s$, and $g(u) = \omega_{n-1}u^{n-1}$ when $u > t$. The John domain properties of Ω_0^c and Ω, respectively, show that there exists $C > 0$ such that

$$g(u) \geqslant C(u - s)^{n-1} \quad \text{and} \quad h(u) \geqslant C(t - u)^{n-1} \quad \text{for every } s \leqslant u \leqslant t. \tag{5.3}$$

We now perform spherical symmetrization on Ω (see [12, pp. 205–210], [8] for more on spherical symmetrization); we obtain Ω' such that $\Omega' \cap S^{n-1}(u)$ is a relatively open spherical cap in $S^{n-1}(u)$, with center $-ue_n$, and $(n-1)$-measure $h(u)$. Then $|\Omega'| = |\Omega|$. Moreover, by [16, Theorem 2.4.2], we may assume that Ω is polyhedral. Then $P(\Omega') \leqslant P(\Omega)$, cf. [8, Lemma 3]. We split the rest of the proof into two cases. We first assume that

$$\Omega' \subset B^{n-1}(t) \times \big(-t, s + 3(r - s)/4\big) =: V. \tag{5.4}$$

We consider the following problem: find a Borel set $E \subset V$ with $|E| = |\Omega'| = |\Omega|$, such that $P(E) \leqslant P(F)$ for every $F \subset V$ with the same properties. Now a solution to this problem exists, and we can apply [15] to conclude that E is convex. Then we are in the setting of Theorem 1.2, so, in particular,

$$\beta(E) \leqslant \varphi\big(\delta(E)\big) \leqslant \varphi\big(\delta\big(\Omega'\big)\big) \leqslant \varphi\big(\delta(\Omega)\big). \tag{5.5}$$

On the other hand, the volume radius of E is r, and, since $E \subset V$, the inradius is at most $(t + s)/2 + 3(r - s)/8$. Thus, as in the proof of Lemma 5.2,

$$\beta(E) \geqslant \frac{2r - t - s - 3(r - s)/4}{2r} \geqslant \frac{r - s}{8r} - C\varphi\big(\delta(\Omega)\big). \tag{5.6}$$

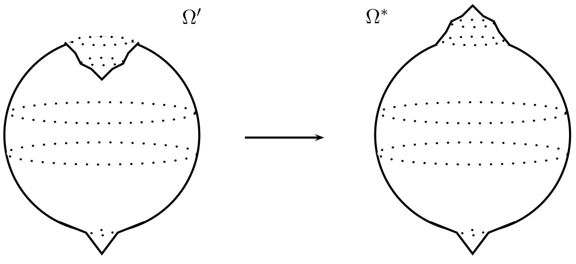

Fig. 8. Reflection.

Combining (5.5) and (5.6) yields $(r-s)/r \leqslant C\varphi(\delta(\Omega))$, which, in view of Lemma 5.2, yields the claim when we assume (5.4).

Next we assume that (5.4) does not hold. Notice that $\Omega' \subset \{-t < x_n\}$. Let $\mu > s + 3(r-s)/4$ be the smallest number such that $\Omega' \subset \{x_n < \mu\}$. Then $\overline{\Omega'} \cap \{x_n = \mu\}$ consists of one or more (possibly infinitely many) $(n-2)$-spheres $S^{n-1}(u) \cap \{x_n = \mu\}$. We denote the smallest such u by v. Then $B^n(v) \cap \{x_n < \mu\} \setminus \overline{\Omega'}$ has a unique component U containing

$$\bigl\{(0, \dots, x_n)\colon s < x_n < \mu\bigr\}.$$

Let U^* be the reflection of U with respect to $\{x_n = \mu\}$, and define Ω^* as the interior of (see Fig. 8)

$$\Omega' \cup \overline{U} \cup U^*.$$

Notice that the boundary points of U inside $B^n(v)$ are interior points of Ω^*, and that the only new boundary in taking the above union comes from the boundary of U^*. Hence $P(\Omega^*) \leqslant P(\Omega')$. Moreover, the circumradius R^* satisfies

$$R^* = \frac{t + 2\mu - s}{2} \geqslant \frac{t + s + 3(r-s)/2}{2} \geqslant r + (r-s)/4,$$

and the volume is

$$\begin{aligned} |\Omega| + 2|U| &\leqslant |\Omega| + 2\bigl|U \cap B^n(r)\bigr| + 2\bigl|B^n(t) \setminus \overline{B^n(r)}\bigr| \\ &\leqslant \alpha_n r^n + 2\alpha_n \lambda(\Omega) r^n + 2\alpha_n\bigl(t^n - r^n\bigr) \leqslant \alpha_n r^n\bigl(1 + C\varphi\bigl(\delta(\Omega)\bigr)\bigr) \end{aligned}$$

by Lemma 5.2 and Theorem 1.1. So the volume radius r^* of Ω^* is at most $r(1 + C\varphi(\delta(\Omega)))$, and

$$\alpha\bigl(\Omega^*\bigr) = \frac{R^* - r^*}{r^*} \geqslant C\frac{r + (r-s)/4 - r - Cr\varphi(\delta(\Omega))}{r}.$$

Therefore,

$$\frac{r-s}{4r} \leqslant \alpha\bigl(\Omega^*\bigr) + C\varphi\bigl(\delta(\Omega)\bigr).$$

Now we can use Theorem 3.1 to estimate $\alpha(\Omega^*)$. Namely, using (5.3), and applying Schwarz and Steiner symmetrizations as in the proof of Theorem 1.4, we obtain a domain Ω_f satisfying the assumptions of Theorem 3.1, such that $2\alpha(\Omega_f) \geqslant \alpha(\Omega^*)$, and $\delta(\Omega_f) \leqslant 2\delta(\Omega^*)$. Combining Theorem 3.1 with the above estimates yields

$$\frac{r-s}{r} \leqslant C\varphi\big(\delta(\Omega)\big),$$

as desired. The proof is complete.

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